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The 2-Part of the Schur Group

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INTRODUCTION

Let the field K be an abelian extension of the rational field Q. The Schur group of K, $S(K)$, consists of those classes in the Brauer group of K which contain an algebra isomorphic to a simple component of a rational group algebra QG for some finite group G.

In a recent paper [4], Janusz determined the possible Hasse invariants of a set of generators for $S(K)_p$, the p-primary part of $S(K)$, where p was an odd prime or p was 2 and the fourth roots of unity were in K. For p an odd prime, this information led to a decomposition for $S(K)_p$. In this paper we shall carry out this project in the case where $p = 2$ and the fourth roots of unity are not in K. Unfortunately $S(K)_2$ does not decompose nicely. Examples of this may be found in [4, 5, or 6].

Let $[A]$ denote the class of the central simple K-algebra A in the Brauer group of K. If φ is a prime of K then $\mathrm{inv}_\varphi[A]$ will mean the Hasse invariant of $[K_\varphi \otimes_K A]$, where K_φ is the completion of K at φ.

It was shown by Benard and Schacher [2] that if m is the index of $[A] \in S(K)$, then ϵ_m, a primitive mth root of unity, is in K. Moreover, if $\sigma \in \mathrm{Gal}(K/Q)$ and $\sigma(\epsilon_m) = \epsilon_m{}^a$, then

$$\mathrm{inv}_\varphi\,[A] \equiv a\,\mathrm{inv}_{\sigma(\varphi)}\,[A] \qquad \mathrm{mod}\ 1$$

for all primes φ of K. An immediate consequence of this result is that the index of $K_\varphi \otimes_K A$ is equal to the index of $K_{\varphi'} \otimes A$, where $[A] \in S(K)$, whenever φ and φ' divide the same rational prime p. This common index is called the p-local index of $[A]$.

Thus if the field K does not contain the fourth roots of unity, then the local indices of an algebra class $[A] \in S(K)_2$ must be either 1 or 2. The Hasse invariants of $[A]$ are thus completely determined by the local indices of $[A]$.

It is known [2] that a class $[A]$ in the Brauer group of a field K is in $S(K)$

if and only if it contains a cyclotomic algebra. That is, a crossed-product algebra of the form

$$(K(\epsilon_m)/K, \alpha) = \sum K(\epsilon_m)u_\sigma, \qquad \sigma \in G,$$

where ϵ_m is a primitive mth root of unity, G is the Galois group of $K(\epsilon_m)/K$, and α is a factor set whose values are roots of unity in $K(\epsilon_m)$. The multiplication in such an algebra is given by

$$u_\sigma x = \sigma(x)u_\sigma,$$

$$u_\sigma u_\tau = \alpha(\sigma, \tau)u_{\sigma\tau},$$

for $x \in K(\epsilon_m)$ and $\sigma, \tau \in G$. We may assume that u_1 commutes with all of the u_σ.

Let $L = Q(\epsilon_n)$ be the smallest cyclotomic field containing K. We may assume that n is either odd or divisible by 4. Let p be any rational prime. It was proved in [5] that $S(K)_p$ is generated by the classes containing algebras of the form $(L(\epsilon_q)/K, \alpha)$ where α has values in the group of p-power roots of unity in $L(\epsilon_q)$, q is either 4 or an odd prime, and q does not divide n. It is the local indices of these algebras which we shall actually compute.

In Section 1 we describe factor sets in a convenient way. In Sections 2 and 3 the calculations of the local indices are made.

1. Description of a Factor Set

In this section we shall consider the following situation.

K is an abelian extension of Q, $\epsilon_4 \notin K$,
$F = Q(\epsilon)$ is a cyclotomic field containing K,
W is the group of 2-power roots of unity in F,
ζ is a generator for W,
2^s is the order of W,
G is the Galois group of F over K,
C is the Galois group of F over $K(\zeta)$.

The purpose of this section is to describe $H^2(W, G)$ in terms of functions $\Psi: C \times C \to \{1, -1\}$ which satisfy

(1) $\Psi(\eta\lambda, \mu) = \Psi(\eta, \mu)\, \Psi(\lambda, \mu)$,
(2) $\Psi(\eta, \lambda\mu) = \Psi(\eta, \lambda)\, \Psi(\eta, \mu)$,
(3) $\Psi(\eta, \eta) = 1$,

for all η, λ, μ in C. Such functions are called skew pairings.

LEMMA 1.1. *There are two possibilities for G/C.*

(1) $G/C \cong \langle\sigma'\rangle$, *where* $\sigma'(\zeta) = \zeta^{2^r-1}$ *for some* r, $1 \leqslant r \leqslant s$.

(2) $G/C \cong \langle\rho'\rangle \times \langle\sigma'\rangle$, *where* $\rho'(\zeta) = \zeta^{-1}$

and $\sigma'(\zeta) = \zeta^{2^r+1}$ *for some* r, $0 < r < s$.

Proof. The group G/C is isomorphic to the Galois group of $K(\zeta)$ over K, and hence is isomorphic to a subgroup of the Galois group of $Q(\zeta)$ over Q. All subgroups of $\mathrm{Gal}(Q(\zeta)/Q)$ which do not fix ϵ_4 have one of the forms described in the lemma.

We shall treat these two different cases separately. First suppose that G/C is cyclic. Pick an element σ in G such that the coset of σ in G/C generates G/C. Note that $\sigma^{2^t} \in C$, where $2^t = 2^{s-r} = |G/C|$. Let $C = \langle\tau_1\rangle \times \langle\tau_2\rangle \times \cdots \times \langle\tau_n\rangle$ be a fixed decomposition of C.

PROPOSITION 1.2. *If* $[\alpha]$ *is a class in* $H^2(G, W)$, *then there is a skew pairing* $\Psi\colon C \times C \to \{1, -1\}$ *associated with* $[\alpha]$. *Moreover the crossed-product algebra* $(F/K, \alpha) \cong \sum Fv_\tau$, $\tau \in G$, *where the following conditions hold.*

(M1) *If* $\eta = \tau_1^{t_1}\tau_2^{t_2} \cdots \tau_k^{t_k}$, *where* $0 \leqslant t_i < |\tau_i|$, *then*

$$v_\eta = (v_{\tau_1})^{t_1}(v_{\tau_2})^{t_2} \cdots (v_{\tau_k})^{t_k};$$

(M2) *For* η, μ *in* C, $\Psi(\eta, \mu) = v_\eta v_\mu v_\eta^{-1} v_\mu^{-1}$;

(M3) $(v_{\tau_i})^{|\tau_i|} = \epsilon_i$, $i = 1, 2,..., k$, *where* ϵ_i *is in* W;

(M4) (a) $v_\sigma{}^c = v_{\sigma^c}$ *for* $0 \leqslant c < 2^t$,

(b) $v_\sigma^{2^t} = v_{\sigma^{2^t}}$ *if* $\sigma^{2^t} \neq 1$,

(c) $v_\sigma^{2^t} = \epsilon_0$, *where* ϵ_0 *is in* $\{1, -1\}$ *if* $\sigma^{2^t} = 1$;

(M5) $v_\sigma v_{\tau_i} v_\sigma^{-1} = \beta_i v_{\tau_i}$, $i = 1, 2,..., k$, *where* β_i *is in* W.

The $\{\epsilon_i, \beta_i\}$ *must satisfy the conditions*:

(M6) $\Psi(\sigma^{2^t}, \tau_i) = \beta_i{}^V$, $i = 1, 2,..., k$, *where* $V = ((2^r - 1)^{2^t} - 1)/(2^r - 2)$;

(M7) $\epsilon_i^{2^r-2} = \beta_i^{|\tau_i|}$, $i = 1, 2,..., k$.

Conversely, if Ψ *is a skew pairing of* $C \times C$ *into* $\{1, -1\}$, $\{\epsilon_i, \beta_i\}$, $1 \leqslant i \leqslant k$, *are elements which satisfy* (M6) *and* (M7), *and* ϵ_0 *is in* $\{1, -1\}$ *if* $\sigma^{2^t} = 1$, *then there is a factor set* α *which is determined by conditions* (M1)–(M5).

Proof. Suppose that $(F/K, \alpha) = \sum Fu_\tau$. Observe that if $v_\tau = w(\tau)u_\tau$ where $w(\tau) \in W$ for all $\tau \in G$, then

$$\begin{aligned} v_\tau v_\eta &= w(\tau)u_\tau\, w(\eta)u_\eta, \\ &= w(\tau)\, \tau(w(\eta))\, \alpha(\tau, \eta)u_{\tau\eta}, \\ &= (w(\tau)\, \tau(w(\eta))/w(\tau\eta))\, \alpha(\tau, \eta)v_{\tau\eta}. \end{aligned}$$

Thus $(F/K, \alpha) \simeq \sum Fv_\tau$ since we have only altered the factor set α by multiplying it by a principal factor set. Clearly all factor sets with values in W which are equivalent to α arise in this way. Thus we must show that the $\{v_\tau\}$ described in the proposition can be obtained by multiplying the u_τ's by elements in W.

First observe that

$$(w(\tau)u_\tau)(w(\eta)u_\eta)(w(\tau)u_\tau)^{-1}(w(\eta)u_\eta)^{-1} = u_\tau u_\eta u_\tau^{-1} u_\eta^{-1}$$

when τ and η are in C and $w(\tau)$ and $w(\eta)$ are in W. Hence the function Ψ defined in (M2) is independent of the representative of $[\alpha] \in H^2(G, W)$ that we pick. That Ψ is a skew pairing follows directly from the commutator identities

(1) $[wy, z] = x[y, z]\, x^{-1}[x, z]$,

(2) $[x, yz] = [x, y]\, y[x, z]\, y^{-1}$, and

(3) $[x, x] = 1$.

Noting that $u_\sigma u_\tau u_\sigma^{-1} u_\tau^{-1} = \beta(\tau)$ is in W for all $\tau \in C$, we see that

$$\begin{aligned}\sigma(\Psi(\tau, \eta)) &= u_\sigma u_\tau u_\eta u_\tau^{-1} u_\eta^{-1} u_\sigma^{-1},\\ &= \beta(\tau)\, u_\tau \beta(\eta)\, u_\eta \beta(\tau)^{-1} u_\tau^{-1} \beta(\eta)^{-1} u_\eta^{-1},\\ &= \Psi(\tau, \eta).\end{aligned}$$

Thus the values of Ψ are in the subgroup of W fixed by σ, namely, $\{1, -1\}$.

Now set $v_\sigma = u_\sigma$ and $v_{\tau_i} = u_{\tau_i}$ for $i = 1, 2, \ldots, k$. If $\eta = \tau_1^{t_1}\tau_2^{t_2} \cdots \tau_k^{t_k}$, where $0 \leqslant t_i < |\tau_i|$, then set $v_\eta = (v_{\tau_1})^{t_1}(v_{\tau_2})^{t_2} \cdots (v_{\tau_k})^{t_k}$. This can be done since $(u_{\tau_1})^{t_1}(u_{\tau_2})^{t_2} \cdots (u_{\tau_k})^{t_k} = wu_\eta$, where $w \in W$ is a product of different values of α. Similarly set $v_{\sigma^c} = v_\sigma{}^c$ for $0 \leqslant c < 2^t$.

Define $\{\epsilon_i, \beta_i\}$, $1 \leqslant i \leqslant k$, by (M3) and (M5). The properties (M6) and (M7) follow immediately; (M7) by raising the equation $v_\sigma v_{\tau_i} v_\sigma^{-1} = \beta_i v_{\tau_i}$ to the $|\tau_i|$th power, and (M6) by raising $\beta_i^{-1} v_\sigma = v_{\tau_i} v_\sigma v_{\tau_i}^{-1}$ to the 2^tth power.

We are left with showing that we can arrange the $\{v_\gamma\}$ so that (M4)(b) and (M4)(c) hold. Let $\sigma^{2^t} = \eta \in C$. There is an element $\lambda \in W$ such that $v_\sigma^{2^t} = \lambda v_\eta$. If $\eta = 1$, we may set $\epsilon_0 = \lambda$ and, since σ must fix λ in this case, (M4)(c) is satisfied.

Suppose now that $\eta = \tau_1^{t_1}\tau_2^{t_2} \cdots \tau_k^{t_k}$. We have

$$\lambda v_\eta = v_\sigma(v_\sigma^{2^t})v_\sigma^{-1} = \sigma(\lambda)\, v_\sigma v_\eta v_\sigma^{-1} = \lambda^{2^r - 1} \prod_{i=1}^{k} \beta_i^{t_i} v_\eta\,.$$

Thus

$$\lambda^{2^r - 2} = \prod_{i=1}^{k} \beta_i^{t_i}.$$

Since every value of Ψ is in $\{\pm 1\}$, there are elements $\omega_{ij} \in W$ such that $\Psi(\tau_i, \tau_j) = \omega_{ij}^{2^{s-1}}$. We may assume that $\omega_{ij} = \omega_{ji}^{-1}$ and that $\omega_{ii} = 1$.

Letting $V = 2^{s-1}X$, we have from (M6) that

$$\beta_i^{2^{s-1}X} = \Psi(\eta, \tau_i) = \prod_{j=1}^{k} \Psi(\tau_j, \tau_i)^{t_j} = \left(\prod_{j=1}^{k} \omega_{ji}^{t_j}\right)^{2^{s-1}}.$$

Every 2^{s-1}th root of unity is a square in W, so there are elements ξ_i in W such that

$$\beta_i = \xi_i^{\,2} \left(\prod_j \omega_{ji}^{t_j}\right)^Y,$$

where $Y = X^{-1} \bmod 2^s$. Thus

$$\begin{aligned}\lambda^{2N} &= \left(\prod_i \xi_i^{t_i}\right)^2 \left(\prod_i \prod_j \omega_{ji}^{t_j t_i}\right)^Y, \\ &= \left(\prod_i \xi_i^{t_i}\right)^2,\end{aligned}$$

where $N = 2^{r-1} - 1$. Let $Z = N^{-1} \bmod 2^s$, then

$$\lambda = \mu^{2^{s-1}} \left(\prod_i \xi_i^{t_i}\right)^Z,$$

for some $\mu \in W$. Set

$$v_\sigma{}^* = \mu^{-Y} v_\sigma \qquad \text{and} \qquad v_{\tau_i}^* = \xi_i{}^Z v_{\tau_i}.$$

We now have

$$\begin{aligned}(v_\sigma{}^*)^{2^t} &= \mu^{-2^{s-1}XY} (v_\sigma)^{2^t} = \mu^{-2^{s-1}} \lambda v_\eta \\ &= \prod_i \xi_i^{t_i Z} v_\eta = v_\eta{}^*.\end{aligned}$$

The $\{v_\eta{}^*\}$ satisfy conditions (M1)–(M7).

For the converse, assume that $\Psi\colon C \times C \to \{1, -1\}$ is a skew pairing and that the elements $\{\epsilon_i, \beta_i\}$ satisfy (M6) and (M7). If $\sigma^{2^t} = 1$, then there is also an element ϵ_0 in $\{1, -1\}$.

By using conditions (M1), (M2), and (M3) we may define an algebra $(F/K(\zeta), \alpha_c) = \sum F v_\tau$, $\tau \in C$. Further, we may extend α_c to a function $\alpha\colon G \times G \to W$, where

$$\begin{aligned}&\alpha(\sigma^a \tau_1^{a_1} \tau_2^{a_2} \cdots \tau_k^{a_k}, \sigma^b \tau_1^{b_1} \tau_2^{b_2} \cdots \tau_k^{b_k}) \\ &\quad = \sigma^a \left(\prod_{i=1}^{k} \beta_i^{-a_i}\right)^{\frac{(2^r-1)^b - 1}{2^r - 2}} \alpha_c(\tau_1^{a_1} \tau_2^{a_2} \cdots \tau_k^{a_k}, \tau_1^{b_1} \tau^{b_2} \cdots \tau_k^{b_k}).\end{aligned}$$

Conditions (M6) and (M7) express the consistency conditions on α required by (M3), (M4)(a), (M4)(b), or (M4)(c), and the fact that σ^{2^t} is in C. This function α is the required factor set.

This completes the proof of the proposition.

COROLLARY 1.3. *For any skew pairing $\Psi: C \times C \to \{1, -1\}$ there is a factor set α on G such that conditions* (M1)–(M7) *are satisfied.*

Proof. The elements $\{\beta_i\}$ and $\{\epsilon_i\}$ can always be found in W.

Now assume that G/C is not cyclic. Pick elements ρ and σ in G such that $\rho(\zeta) = \zeta^{-1}$ and $\sigma(\zeta) = \zeta^{2^r+1}$. Observe that ρ^2 is in C and σ^{2^t} is in C, where $t = s - r$.

LEMMA 1.4. *The element $\rho \in G$ may be picked so that either $\rho^2 = 1$ or $\rho^2 = \tau_1$, where $C = \langle\tau_1\rangle \times \cdots \times \langle\tau_k\rangle$ is a decomposition of C.*

Proof. Let $C = \langle\sigma_1\rangle \times \cdots \times \langle\sigma_k\rangle$ be any decomposition of C into cyclic groups of prime power order. Pick an element ρ_0 in G such that $\rho_0(\zeta) = \zeta^{-1}$. Now $\rho_0{}^2 = \sigma_1^{t_1}\sigma_2^{t_2} \cdots \sigma_k^{t_k}$ is in C. If the order of σ_i is $p_i^{a_i}$ where p_i is odd, then let r_i be an integer satisfying $2r_i \equiv -t_i \bmod p_i^{a_i}$. If the order of σ_i is 2^{a_i} and t_i is even, then let $r_i = -t_i/2$. In the other cases let $r_i = 0$. Set $\rho = \rho_0 \cdot \sigma_1^{r_1}\sigma_2^{r_2} \cdots \sigma_k^{r_k}$. If $\rho^2 \neq 1$, let $\max\{| \sigma_i^{t_i+2r_i} |\} = \sigma_j^{t_j}$. Observe that $| \sigma_j^{t_j} | = | \sigma_j |$, so we have that

$$C = \langle\rho^2\rangle \times \langle\sigma_1\rangle \times \cdots \times \langle\sigma_{j-1}\rangle \times \langle\sigma_{j+1}\rangle \times \cdots \times \langle\sigma_k\rangle.$$

Fix a decomposition $C = \langle\tau_1\rangle \times \cdots \times \langle\tau_k\rangle$ of C where $\rho^2 = \tau_1$. We now prove a proposition similar to Proposition 1.2.

PROPOSITION 1.5. *If $[\alpha]$ is a class in $H^2(G, W)$, then there is a skew pairing $\Psi: C \times C \to \{1, -1\}$ associated with $[\alpha]$. Moreover, the crossed-product algebra $(F/K, \alpha) \cong \sum Fv_\tau$, $\tau \in G$, where the following conditions hold.*

(N1) *If $\eta = \tau_1^{t_1}\tau_2^{t_2}\cdots\tau_k^{t_k}$, where $0 \leqslant t_i < |\tau_i|$, then $v_\eta = (v_{\tau_1})^{t_1}(v_{\tau_2})^{t_2}\cdots(v_{\tau_k})^{t_k}$;*

(N2) *For η, λ in C, $\Psi(\eta, \lambda) = v_\eta v_\lambda v_\eta^{-1} v_\lambda^{-1}$;*

(N3) *$(v_{\tau_i})^{|\tau_i|} = \epsilon_i$, $i = 1, 2,\ldots, k$, where ϵ_i is in W;*

(N4) (a) *$v_\sigma{}^c = v_{\sigma^c}$ for $0 \leqslant c < 2^t$;*

(b) *$v_\rho{}^2 = \epsilon_*$ if $\rho^2 = 1$, where ϵ_* is in $\{1, -1\}$, $v_\sigma^{2^t} = \epsilon_0$ if $\sigma^{2^t} = 1$, where ϵ_0 is in W and is fixed by σ;*

(c) *$v_\rho{}^2 = v_{\rho^2}$ if $\rho^2 \neq 1$, $v_\sigma^{2^t} = v_{\sigma^{2^t}}$ if $\sigma^{2^t} \neq 1$;*

(N5) (a) *$v_\rho v_{\tau_i} v_\rho^{-1} = \gamma_i v_{\tau_i}$, $i = 1, 2,\ldots, k$, where γ_i is in W;*

(b) *$v_\sigma v_{\tau_i} v_\sigma^{-1} = \delta_i v_{\tau_i}$, $i = 1, 2,\ldots, k$, where δ_i is in W;*

(c) *$v_\sigma v_\rho v_\sigma^{-1} = \delta_0 v_\rho$, where δ_0 is in W.*

The $\{\epsilon_i, \gamma_i, \delta_i\}$ *must satisfy the conditions*:

(N6) (a) $\Psi(\rho^2, \tau_i) = 1$ *for* $i = 1, 2, \ldots, k$;

(b) $\Psi(\sigma^{2^t}, \tau_i) = \delta_i^U$ *for* $i = 1, 2, \ldots, k$, *where* $U = ((1 + 2^r)^{2^t} - 1)/2^r$;

(N7) (a) $\epsilon_i^{-2} = \gamma_i^{|\tau_i|}$ *for* $i = 1, 2, \ldots, k$;

(b) $\epsilon_i^{2^r} = \delta_i^{|\tau_i|}$ *for* $i = 1, 2, 3, \ldots, k$;

(N8) (a) $\delta_1 = 1$ *if* $\rho^2 \neq 1$;

(b) $\delta_0^{-U} = \prod_{i=1}^k \gamma_i^{a_i}$, *where* $\sigma^{2^t} = \tau_1^{a_1}\tau_2^{a_2} \cdots \tau_k^{a_k} \neq 1$;

(c) $\delta_0^U = \epsilon_0$, *where* $\sigma^{2^t} = 1$;

(N9) $\delta_i^2 = \gamma_i^{-2^r}$ *for* $i = 1, 2, \ldots, k$.

Conversely, if Ψ *is a skew pairing* $C \times C$ *into* $\{1, -1\}$, $\{\epsilon_i, \gamma_i, \delta_i\}$ $1 \leqslant i \leqslant k$, *are elements which satisfy* (N6)–(N9), ϵ_* *is in* $\{1, -1\}$ *if* $\rho^2 = 1$, *and* ϵ_0 *is in* W *fixed by* σ *if* $\sigma^{2^t} = 1$, *then there is a factor set* α *determined by conditions* (N1)–(N5).

Proof. The proof of this proposition is essentially the same as that of Proposition 1.2. The only point where extra care is required is in showing that (N4)(c) is satisfied.

Let $(F/K, \alpha) \simeq \sum Fu_\tau$, where the $\{u_\tau\}$ satisfies (N1)–(N3) and (N5)–(N7). Suppose that $u_\rho^2 = \lambda_* u_{\tau_1}$ and that $u_\sigma^{2^t} = \lambda u_\eta$, where $\sigma^{2^t} = \eta = \tau_1^{a_1}\tau_2^{a_2} \cdots \tau_k^{a_k}$ and λ_* and λ are in W. As before we get

$$\lambda\sigma(\lambda)^{-1} = \lambda^{-2^r} = \prod_{i=1}^{k} \delta_i^{a_i}.$$

Let $\{\omega_{ij}\}$, $1 \leqslant i, j \leqslant k$, be elements in W which satisfy

$$\Psi(\tau_i, \tau_j) = \omega_{ij}^{2^t}.$$

We may assume that $\omega_{ij} = \omega_{ji}^{-1}$ and that $\omega_{ii} = 1$. Further, (N6)(a) allows us to assume that $\omega_{1i} = 1$ for all i.

Set $U = 2^t X$. We get

$$\delta_i^{2^t X} = \Psi(\eta, \tau_i) = \prod_j \Psi(\tau_j, \tau_i)^{a_j} = \left(\prod_j \omega_{ji}^{a_j}\right)^{2^t}.$$

Thus for some elements ξ_i in W we get

$$\delta_i = \xi_i^{2^r}\left(\prod_j \omega_{ji}^{a_j}\right)^Y$$

where $Y = X^{-1} \bmod 2^s$. In the case where $i = 1$ we may pick $\xi_1 = \lambda_*^{-1}$ since (N5)(c) implies that

$$\delta_1 = \lambda_* \sigma(\lambda_*)^{-1} = \lambda_*^{-2^r}.$$

We now have that

$$\lambda^{-2^r} = \left(\prod_i \xi_i^{a_i}\right)^{2^r} \left(\prod_i \prod_j \omega_{ji}^{a_i a_j}\right)^Y = \left(\prod_i \xi_i^{a_i}\right)^{2^r}.$$

Hence there is an element μ in W such that

$$\lambda = \mu^{2^t} \prod_i \xi_i^{-a_i}.$$

Set

$$v_\sigma = \mu^{-Y} u_\sigma \qquad \text{and} \qquad v_{\tau_i} = \xi_i^{-1} u_{\tau_i}.$$

This choice of $\{v_\tau\}$ satisfies (N4)(c). Condition (N8) now follows from (N5)(c), and (N9) follows from the associativity of the multiplication of v_ρ, v_σ, and v_{τ_i}.

The proof of the converse is the same as in Proposition 1.2.

COROLLARY 1.6. *For any skew pairing* $\Psi: C \times C \to \{1, -1\}$ *there is a factor set* α *on* G *such that conditions* (N1)–(N9) *are satisfied.*

2. THE p-LOCAL INDICES WHERE p IS ODD

In this section we shall compute the possible p-local indices of elements in $S(K)_2$ for all odd primes p.

We shall keep the notation of the preceding section, and add the following, rather extensive, list to it.

$L = Q(\epsilon_n)$ is the least cyclotomic field containing K,
q is either 4 or an odd prime,
$F = L(\epsilon_q)$,
p is an odd prime dividing nq,
φ is a prime of K dividing p,
$f = f(p)$ is the residue class degree of φ over p,
2^g is the power of 2 dividing $p^{f(p)} - 1$,
$\theta = \theta(p)$ is a generator of the inertia group of φ in $G = \mathrm{Gal}(F/K)$;
$e = e(p)$ is the order of θ;
2^h is the power of 2 dividing $e(p)$;
$\nu(p) = \max\{0, 1 + h - g\}$.

If G/C is cyclic then

σ is an element of G whose coset modulo C generates G/C;
$\sigma(\zeta) = \zeta^{2^{r-1}}$ for some integer r, $1 \leqslant r \leqslant 2^s$;
σ^{2^t} is in C;
$\phi = \phi(p)$ is a Frobenius automorphism of φ in G;
$\phi = \sigma^{a(p)}\xi_p$, where ξ_p is in C;
$V(x) = ((2^r - 1)^x - 1)/(2^r - 2)$ for any integer x;
Y is an integer satisfying $Y(2^{r-1} - 1) \equiv 1 \bmod 2^s$;
$Z(p) = (Y(p^{f(p)} - 1)/2) - V(a(p))$;
$X(p)$ is an integer satisfying $X(p)\, V(2^t) \equiv Z(p) \bmod 2^s$;
$\eta(p) = \sigma^{-2^t X(p)}\xi_p$ is an element in G.

If G/C is not cyclic then

ρ is an element in G such that $\rho(\zeta) = \zeta^{-1}$;
σ is an element in G such that $\langle \rho C\rangle \times \langle \sigma C\rangle \cong G/C$;
$\sigma(\zeta) = \zeta^{2^r+1}$ for some integer r, $0 < r < s$;
σ^{2^t} is in C;
$\phi = \phi(p)$ is a Frobenius automorphism of φ in G;
$\phi = \rho^{a'(p)}\,\sigma^{a(p)}\xi_p$, where ξ_p is in C;
$V(x) = ((2^r + 1) - 1)/2^r$ for any integer x;
Y is an integer satisfying $Y(2^{r-1} + 1) \equiv 1 \bmod 2^{s-r}$;
$Z(p) = (p^{f(p)} - (-1)^{a'(p)}(1 + 2^r)^{a(p)})2^{-r}$;
$X(p)$ is an integer satisfying $X(p)\, V(2^t) \equiv Z(p) \bmod 2^s$;
$\eta(p) = \sigma^{-2^t X(p)}\xi_p$ is an element in G.

THEOREM 2.1. *Suppose that α is a factor set on G with values in W. Then the p-local index of the class $[(F/K, \alpha)]$ is the order of*

$$\Phi_p(\alpha) = (\alpha(\theta, \phi)/\alpha(\phi, \theta))(\epsilon_\theta)^{(p^f-1)/e},$$

where $\epsilon_0 = \alpha(\theta, \theta)\, \alpha(\theta^2, \theta) \cdots \alpha(\theta^{e-1}, \theta)$.

Proof. This is essentially Yamada's Theorem 4.3 [7].

THEOREM 2.2. *Let α be a factor set on G with values in W, and let Ψ be the skew pairing from $C \times C$ into $\{-1, 1\}$ determined by α. Then*

$$\Phi_p(\alpha) = \Psi(\theta(p), \eta(p))\, \mu_p(\alpha),$$

where $\mu_p(\alpha)$ is a root of unity of order dividing $2^{\nu(p)}$. As α runs through all factor sets with values in W which determine Ψ, the element $\mu_p(\alpha)$ runs through all the $2^{\nu(p)}$th roots of unity.

Proof. Let $(F/K, \alpha) = \sum Fv_\tau$, $\tau \in G$, where $\{v_\tau\}$ satisfy either conditions (M1)–(M7) or conditions (N1)–(N9). Since p is an odd prime, $\theta(p)$ is in C. Set $\theta(p) = \tau_1^{t_1}\tau_2^{t_2} \cdots \tau_k^{t_k}$. Then

$$\epsilon_\theta = (v_\theta)^e = \mu_0 \prod_{i=1}^{k} \epsilon_i^{t_i e/|\tau_i|},$$

where μ_0 is either 1 or -1 since the values of Ψ are in $\{-1, 1\}$. Hence

$$\begin{aligned}\Phi_p(\alpha) &= (\alpha(\theta, \phi)/\alpha(\phi, \theta))(\epsilon_\theta)^{(p^{f(p)}-1)/e(p)} \\ &= (v_\theta v_\phi v_\theta^{-1} v_\phi^{-1}) \left(\prod_{i=1}^{k} \epsilon_i^{T_i}\right) \mu_0^{(p^{f(p)}-1)/e(p)},\end{aligned}$$

where $T_i = t_i(p^{f(p)} - 1)/|\tau_i|$.

Now for at least one value of i, $|\tau_i| = et_i$. Let j be one such value. Then $T_j = (p^{f(p)} - 1)/e(p)$ and so the least nonnegative integer ν such that $2^\nu T_j$ is even is $\nu(p) = \max\{0, 1 + h - g\}$. Hence $\nu(p)$ is the least nonnegative integer such that $2^{\nu(p)}T_i$ is even for all $i = 1, 2, 3,\ldots, k$.

Step 1. When G/C is cyclic, $\Phi_p(\alpha) = \Psi(\theta, \eta)\, \mu_p(\alpha)$.

From condition (M7) we get

$$\begin{aligned}\epsilon_i^{2^{\nu(p)}T_i} &= (\epsilon_i^{2^r-2})^{2^{\nu(p)}T_i Y/2} \\ &= (\beta_i^{|\tau_i|})^{2^{\nu(p)}T_i Y/2} \\ &= (\beta_i^{t_i})^{2^{\nu(p)}Y(p^{f(p)}-1)/2}.\end{aligned}$$

Thus

$$\prod_{i=1}^{k} \epsilon_i^{T_i} = \left(\prod_{i=1}^{k} \beta_i^{t_i}\right)^{Y(p^{f(p)}-1)/2} \mu_1, \tag{1}$$

where μ_1 is a $2^{\nu(p)}$th root of unity. Set $\mu_p(\alpha) = \mu_1(\mu_0)^{(p^{f(p)}-1)/e(p)}$. We now have

$$\Phi_p(\alpha) = (v_\theta v_\phi v_\theta^{-1} v_\phi^{-1}) \left(\prod \beta_i^{t_i}\right)^{Y(p^{f(p)}-1)/e(p)} \mu_p(\alpha).$$

Now $\phi = \sigma^{a(p)}\xi_p$, so

$$\begin{aligned}v_\theta v_\phi v_\theta^{-1} v_\phi^{-1} &= v_\theta v_\sigma^{a(p)} v_\theta^{-1} v_\theta v_{\xi_p} v_\theta^{-1} v_\phi^{-1} \\ &= \left(\prod_{i-1}^{k} \beta_i^{-t_i}\right)^{V(a(p))} \Psi(\theta, \xi_p).\end{aligned}$$

Hence

$$\Phi_p(\alpha) = \left(\prod \beta_i^{t_i}\right)^{Z(p)} \Psi(\theta, \xi_p)\, \mu_p(\alpha).$$

By the definition of the Frobenius automorphism, $p^{f(p)} \equiv (2^r - 1)^{a(p)} \bmod 2^s$. Hence 2^{s-1} divides $Z(p)$ and so there exists an integer $X(p)$ such that $X(p)\, V(2^t) \equiv Z(p) \bmod 2^s$. Thus

$$\begin{aligned}\Big(\prod_i \beta_i^{t_i}\Big)^{Z(p)} &= \Big(\prod_i \beta_i^{V(2^t)t_i}\Big)^{X(p)} \\ &= \Big(\prod_i \Psi(\sigma^{2^t}, \tau_i^{t_i})^{X(p)}\Big) \\ &= \Psi(\sigma^{2^t X(p)}, \theta(p)).\end{aligned}$$

So finally

$$\Phi_p(\alpha) = \Psi(\theta(p), \eta(p))\, \mu_p(\alpha).$$

Step 2. When G/C is not cyclic, $\Phi_p(\alpha) = \Psi(\theta(p), \eta(p))\, \mu_p(\alpha)$. From the equation $v_\rho v_\sigma v_{\tau_i} v_\sigma^{-1} v_\rho^{-1} = \beta_i^{-1}\gamma_i v_{\tau_i}$ we get the relation

$$\epsilon_i^{2^r+2} = \beta_i^{|\tau_i|}\gamma_i^{-|\tau_i|}.$$

Thus

$$\begin{aligned}\epsilon_i^{2^{\nu(p)}\Gamma_i} &= (\epsilon_i^{2^r+2})^{2^\nu \Gamma_i Y/2} \\ &= (\beta_i^{t_i}\gamma_i^{-t_i})^{2^{\nu(p)}Y(p^{f(p)}-1)/2}.\end{aligned}$$

So

$$\prod_i \epsilon_i^{\Gamma_i} = \Big(\prod_i \beta_i^{t_i}\Big)^{Y(p^{f(p)}-1)/2} \Big(\prod_i \gamma_i^{-t_i}\Big)^{Y(p^{f(p)}-1)/2} \mu_1, \tag{2}$$

where μ_1 is a $2^{\nu(p)}$th root of unity. Set $\mu^* = \mu_1(\mu_0)^{(p^{f(p)}-1)/e(p)}$. Further,

$$u_\theta u_\phi u_\theta^{-1} u_\phi^{-1} = \Big(\prod \gamma_i^{-t_i}\Big)^{a'(p)} \Big(\prod \beta_i^{-t_i}\Big)^{(-1)^{a'(p)}V(a(p))} \Psi(\theta, \xi_p).$$

Hence

$$\Phi_p(\alpha) = \Big(\prod \gamma_i^{-t_i}\Big)^{\Gamma + a'(p)} \Big(\prod \beta_i^{t_i}\Big)^{\Gamma - (-1)^{a'(p)}V(a(p))} \Psi(\theta, \xi_p)\mu^*,$$

where $\Gamma = Y(p^{f(p)} - 1)/2$.

From the definition of a Frobenius automorphism we get that $p^{f(p)} \equiv (-1)^{a'(p)}(1 + 2^r)^{a(p)} \bmod 2^s$. Hence $2^{\nu(p)}(\Gamma + a'(p)) \equiv 0 \bmod 2^{r-1}$. Thus from (N9) we get

$$\Big(\prod \gamma_i^{-t_i}\Big)^{\Gamma + a'(p)} = \Big(\prod \beta_i^{t_i}\Big)^{2(\Gamma + a'(p))/2^r} \mu,$$

where μ is a $2^{\nu(p)}$th root of unity. Set $\mu_p(\alpha) = \mu^*\mu$. Now

$$\begin{aligned}\Phi_p(\alpha) &= \left(\prod \beta_i^{t_i}\right)^{1/2^r(p^{f(p)}-1-(-1)^{a'(p)}(1+2^r)^{a(p)}+1)} \Psi(\theta, \xi_p)\, \mu_p(\alpha) \\ &= \left(\prod \beta_i^{t_i}\right)^{Z(p)} \Psi(\theta, \xi_p)\, \mu_p(\alpha) \\ &= \Psi(\theta(p), \eta(p))\, \mu_p(\alpha).\end{aligned}$$

Step 3. Every $2^{\nu(p)}$th root of unity can occur as a value of $\mu(\alpha)$ for some factor set α which determines Ψ.

If $h \neq g$, then $\nu(p) = 0$ and we are done. So assume that $h = g$. Let j be such that $|\tau_j| = e(p)t_j$. Then $T_j = p^{f(p)} - 1/e(p)$ is odd. Switch ϵ_j to $-\epsilon_j$, and call the new factor set α'. Observe that α' does determine Ψ. Then from Eqs. (1) and (2) we observe that $\mu_p(\alpha) = -\mu_p(\alpha')$.

This completes the proof of the theorem.

We shall now use this theorem to calculate the possible local indices of $S(K)_2$.

DEFINITION. (a) Let $N(p)$ denote the order of $\eta(p)$ in C/C^2.

(b) Let $N_1(p)$ denote the order of $\theta(p)\,\eta(p)$ in C/C^2.

(c) Let $N_2(p)$ denote the order of $\theta(p)^{f(p)}$ in C/C^2.

THEOREM 2.3. (A) *If p does not divide n, then the maximum p-local index of an element in $S(K)_2$ is* $\max\{2^{\nu(p)}, N(p)\}$.

The cyclic algebra $D(p) = (K(\epsilon_p), \tau, -1)$ where $\langle\tau\rangle = \mathrm{Gal}(K(\epsilon_p)/K)$ has p-local index $2^{\nu(p)}$ and l-local index 1 for all primes $l \neq p$.

The crossed-product algebra $B(p, \alpha) = (L(\epsilon_q)/K, \alpha)$ has p-local index $N(p)$ if the skew pairing Ψ associated with α is such that $\Psi(\theta(p), \eta(p))$ has order $N(p)$ and $\mu_p(\alpha)$ has order less than $N(p)$.

(B) *If p does divide n, then the maximum p-local index of an element in $S(K)_2$ is* $\max\{2^{\nu(p)}, N_1(p), N_2(p)\}$.

The cyclic algebra $D(p) = (K(\epsilon_p), \theta, -1)$ has p-local index $2^{\nu(p)}$ and l-local index 1 for all primes $l \neq p$.

The crossed-product algebra $A(\alpha) = (L/K, \alpha)$, made with a factor set α on $\mathrm{Gal}(L/K)$ *having values in W, has p-local index $N_1(p)$ if the pairing Ψ associated with α has the property that $\Psi(\theta, \sigma^{-2^t X(p)}\xi_p)$ has order $N_1(p)$ and $\mu_p(\alpha)$ has order less than $N_1(p)$.*

The crossed product $B(p, q, \alpha) = (L(\epsilon_q)/K, \alpha)$ has p-local index $N_2(p)$ if

(1) *α is a factor set on G with values in W and the pairing Ψ associated with α satisfies the three conditions*

(a) $\Psi(\tau, \gamma) = 1$ *for* $\tau, \gamma \in C \cap \mathrm{Gal}(L/K)$;

(b) $\Psi(\theta, \tau^{f(p)})$ *has order* $N_2(p)$, *where* $\langle\tau\rangle = \mathrm{Gal}(L(\epsilon_q)/L)$;

(c) $\mu_p(\alpha)$ *has order less than* $N_2(p)$; *and*

(2) *q is an odd prime such that p is not a square modulo q.*

Proof. Once we have the results of Theorem 2.2, this is exactly the same as [4, Theorems 3 and 5].

The results of this section are very satisfying, for they show that for odd primes p, the p-local indices of elements in $S(K)$ can be calculated in the same way, regardless of which field K is being considered. The only exceptions to these rules occur at the 2-local indices of elements in $S(K)$. These will be considered in the next section.

3. The 2-Local Indices

In this section we compute the possible 2-local indices of elements in $S(K)_2$.

We shall keep the notation of Section 1 and fix the following.

$L = \underset{\sim}{Q}(\epsilon_n)$ is the least cyclotomic field containing K;
q is either 4 or an odd prime;
$F = L(\epsilon_q)$;
φ is a prime divisor of 2 in K;
$\theta = \theta(2)$ is a generator of the inertia group of φ in G;
$\phi = \phi(2)$ is a Frobenius automorphism of φ in G;
$\phi = \sigma^x\xi$, where ξ is in C (let $x = 0$ if G/C is cyclic);
$f = f(2)$ is the residue class degree of φ over 2;
2^g is the power of 2 dividing $f(2)$.

Theorem 3.1. *Suppose that α is a factor set on G with values in W. Then the 2-local index of the class $[(F/K, \alpha)]$ is 1 if $\theta(\zeta) \neq \zeta^{-1}$ and is equal to the order of*

$$\Phi_2(\alpha) = (\alpha(\theta, \phi)/\alpha(\phi, \theta))^{2^{g-1}}(\alpha(\theta, \theta))^{2^{s-2}f}$$

if $\theta(\zeta) = \zeta^{-1}$.

Proof. If $\theta(\zeta) \neq \zeta^{-1}$, then $S(K)_2$ is trivial by either [7, Theorem 5.3] or [3, Theorem 5]. For the remainder of the proof we shall assume that $\theta(\zeta) = \zeta^{-1}$.

Let $(F/K, \alpha) = \sum Fu_\tau$, and set

$$\zeta^\lambda = u_\theta u_\phi u_\theta^{-1} u_\phi^{-1} = \frac{\alpha(\theta, \phi)}{\alpha(\phi, \theta)}.$$

If λ is odd, then

$$u_\theta(1+\zeta^\lambda)u_\phi = \theta(1+\zeta^\lambda)\zeta^\lambda\, u_\phi u_\theta\,,$$
$$= (1+\zeta^\lambda)u_\phi u_\theta\,.$$

If λ is even, then

$$u_\theta \zeta^{\lambda/2} u_\phi = \theta(\zeta^{\lambda/2})\zeta^\lambda\, u_\phi u_\theta = \zeta^{\lambda/2} u_\phi u_\theta\,.$$

Set

$$\begin{aligned}\delta &= 1+\zeta^\lambda && \text{if } \lambda \text{ is odd,}\\ &= \zeta^{\lambda/2} && \text{if } \lambda \text{ is even.}\end{aligned}$$

Let K_φ be the completion of K at φ, and let R be the subfield of $K_\varphi \otimes_k F$ fixed by θ.

We have

$$[K_\varphi \otimes (F/K, \alpha)] = \left[\sum_i K_\varphi(\epsilon_n)\, u_\theta{}^i\right] \otimes \left[\sum_j R(\delta u_\phi)^j\right]$$
$$= [(K_\varphi(\epsilon_4), \theta, \alpha(\theta, \theta))] \otimes [(R, \phi, (\delta u_\phi)^{|\phi|})],$$

where $(K_\varphi(\epsilon_4), \theta, \alpha(\theta, \theta))$ and $(R, \phi, (\delta u_\phi)^{|\phi|})$ are cyclic crossed product algebras. We have identified θ and ϕ with their restrictions to $K_\varphi(\epsilon_4)$ and R, respectively.

Let $\langle \rho \rangle = \mathrm{Gal}(Q_2(\epsilon_4)/Q_2)$. Then

$$[(K_\varphi(\epsilon_4), \theta, \alpha(\theta, \theta))] = K_\varphi \otimes [(Q_2(\epsilon_4), \rho, \alpha(\theta, \theta))].$$

Since the index of $[(Q_2(\epsilon_4), \rho, \alpha(\theta, \theta))]$ is the order of $\alpha(\theta, \theta)$, the index of $[(K_\varphi(\epsilon_4), \theta, \alpha(\theta, \theta))]$ is the order of $(\alpha(\theta, \theta))^{[K_\varphi : Q_2]} = (\alpha(\theta, \theta))^{2^{s-2}f}$.

Now let V and V' be the valuations in K_φ and $K_\varphi(\epsilon_4)$ respectively. Then

$$V((\delta u_\phi)^{|\phi|}) = 1/2V'((\delta u_\phi)^{|\phi|})$$
$$= 1/2 \sum_{j=0}^{|\phi|-1} V'(\phi^j(\delta)),$$

since $u_\phi^{|\phi|}$ is a unit. If λ is odd, then $\phi^j(\delta)$ is a prime element in $K_\varphi(\epsilon_4)$ for all j. So in this case the invariant of $[R, \phi, (\delta u_\phi)^{|\phi|})]$ is $(|\phi|/2)/|\phi| = \frac{1}{2}$. Hence the index is 2, which is the order of $(\zeta^\lambda)^{2^{s-1}}$. If λ is even, then $\phi^j(\delta)$ is a unit in $K_\varphi(\epsilon_4)$ for all j. So in this case $[(R, \phi, (\delta u_\phi)^{|\phi|})]$ has index 1, which is again the order of $(\zeta^\lambda)^{2^{s-1}}$.

This completes the proof of the theorem.

We now need to discover exactly when a factor set α such that $\Phi_2(\alpha)$ has order 2 exists.

THEOREM 3.2. *There exists a factor set α on G with values in W such that $\Phi_2(\alpha)$ has order 2 if and only if $\theta(\zeta) = \zeta^{-1}$ and one of the following occurs.*

(1) $2^{s-2}f(2)$ *is odd,*

(2) G/C *is cyclic and* ϕ *is not a square in* C,

(3) G/C *is not cyclic and* x *is odd, or*

(4) G/C *is not cyclic and* ξ, σ^{2^t}, *and* $\xi\sigma^{2^t}$ *are not squares in* C.

Proof. If $\theta(\zeta) \neq \zeta^{-1}$, then $S(K)_2$ is trivial. So we shall assume that $\theta(\zeta) = \zeta^{-1}$ throughout the proof.

First assume that $2^{s-2}f(2)$ is odd. Then, picking α so that $\alpha(\theta, \theta) = -1$ is the only nontrivial image of α, we get $\Phi_2(\alpha) = -1$.

Now suppose that G/C is cyclic and ϕ is not a square in C. We are in the situation of Proposition 1.2 with $\sigma = \theta$, $\sigma^2 = 1$, and $r = s$. So 2^s divides V which means that condition (M6) places no limitations on any of the β_i. Let $\phi = \tau_1^{t_1}\tau_2^{t_2} \cdots \tau_k^{t_k}$. At least one of the t_i is odd, say t_j. Then set $\beta_j = \zeta$ and $\epsilon_j = \zeta^{Y|\tau_j|/2}$, where $Y(2^{r-1} - 1) \equiv 1 \bmod 2^s$. Let β_i and ϵ_i be 1 for $i \neq j$, and set $\Psi(\eta, \mu) = 1$ for all η, μ in C. Then

$$\alpha(\theta, \phi)/\alpha(\phi, \theta) = v_\theta v_\phi v_\theta^{-1} v_\phi^{-1} = \prod_{i=1}^{k} \beta_i^{t_i} = \zeta^{t_j}.$$

Hence $\Phi_2(\alpha) = (\zeta^{t_j})^{2^{s-1}} = -1$.

Now suppose that G/C is not cyclic. We are in the situation of Proposition 1.5 with $\rho = \theta$. Let $\phi = \sigma^x\tau_1^{t_1}\tau_2^{t_2} \cdots \tau_k^{t_k}$ and $\sigma^{2^t} = \tau_1^{a_1}\tau_2^{a_2} \cdots \tau_k^{a_k}$. We may assume that 2^t does not divide a_i for any i, since otherwise we could replace σ with $\sigma\tau_i^{a_i/2^t}$.

Assume that x is odd. If $\sigma^{2^t} = 1$, then set $\epsilon_0 = \zeta^U$ and $\gamma_i = 1$ for $i = 1, 2, \ldots, k$. Otherwise set $\gamma_j^{a_j} = \zeta^{-U}$ for some j such that $a_j \neq 0$, and set $\gamma_i = 1$ for $i \neq j$. This can be done since 2^t divides U. Observe that in either case each γ_i has order less than 2^s. Now set $\delta_0 = \zeta$ and condition (N8) is satisfied. Let Ψ be the skew pairing such that $\Psi(\eta, \mu) = 1$ for all η, μ in C. Now define elements δ_i, ϵ_i for $i = 1, 2, \ldots, k$ so that (N6), (N7), and (N9) are satisfied. By Proposition 1.5, this system determines a factor set α on G with values in W. Further, we have

$$\alpha(\theta, \phi)/\alpha(\phi, \theta) = v_\theta v_\phi v_\theta^{-1} v_\phi^{-1} = \delta_0{}^x \prod \gamma_i^{t_i}.$$

Hence $\Phi_2(\alpha) = (\delta_0{}^x)^{2^{s-1}} = -1$.

Now assume that neither ξ, nor σ^{2^t}, nor $\xi\sigma^{2^t}$ is a square in C. Then for some j and l, $a_j t_l + a_l t_j$ is odd. Suppose that a_j and t_l are odd. Then set $\gamma_l = \zeta$ and $\gamma_j = \zeta^y$, where $y a_j \equiv -a_l \bmod 2^s$. For $i \neq j$ or l set $\gamma_i = \delta_i = \epsilon_i = 1$. Further, let $\Psi(\tau_j, \tau_l) = \Psi(\tau_l, \tau_j) = -1$ and $\psi(\tau_i, \eta) = 1$ for $i \neq j$

and η in C. Set $\delta_j = \gamma_j^{-2^{r-1}}$, $\delta_l = \gamma_l^{-2^{r-1}}$, $\epsilon_j = \gamma_j^{-|\tau_j|/2}$ and $\epsilon_l = \gamma_l^{-|\tau_l|/2}$. This system satisfies conditions (N6), (N7), (N8), and (N9) so it determines a factor set α on G with values in W. We have

$$\alpha(\theta, \phi)/\alpha(\phi, \theta) = v_\theta v_\phi v_\theta^{-1} v_\phi^{-1} = \delta_0{}^x \prod \gamma_i^{t_i} = \gamma_j^{t_j}\gamma_l^{t_l}.$$

Now by the construction, $\gamma_j^{t_j}$ has order less than 2^s and $\gamma_l^{t_l}$ has order 2^s. Hence $\Phi_2(\alpha) = (\gamma_j^{t_j}\gamma_l^{t_l})^{2^{s-1}} = -1$.

Now let α be any factor set on G with values in W. Since $\alpha(\theta, \theta)$ is fixed by θ, $\alpha(\theta, \theta)$ is in $\{1, -1\}$. Hence if $2^{s-2}f$ is even, $(\alpha(\theta, \theta))^{2^{s-2}f} = 1$.

If G/C is cyclic and ϕ is a square in C, then

$$(\alpha(\theta, \phi)/\alpha(\phi, \theta))^{2^{s-1}} = \left(\prod_{i=1}^{k} \beta_i^{t_i}\right)^{2^{s-1}} = \left(\prod_{i=1}^{k} \beta_i^{t_i/2}\right)^{2^s} = 1.$$

If G/C is not cyclic, x is even, and ξ is a square in C, then

$$(\alpha(\theta, \phi)/\alpha(\phi, \theta))^{2^{s-1}} = \left(\delta_0{}^x \prod_{i=1}^{k} \gamma_i^{t_i}\right)^{2^{s-1}} = \left(\delta_0^{x/2} \prod_{i=1}^{k} \gamma_i^{t_i/2}\right)^{2^s} = 1.$$

Now suppose that G/C is not cyclic, x is even, and σ^{2^t} is a square in C. We have

$$\Psi(\sigma^{2^t}, \tau_i) = \left(\prod_{j=1}^{k} \Psi(\tau_j^{a_i/2}, \tau_i)\right)^2 = 1.$$

Hence $1 = \delta_i^{-2^t} = \gamma_i^{2^{r+t-1}}$, $i = 1, 2,\ldots, k$. But $r + t - 1 = s - 1$, so

$$(\alpha(\theta, \phi)/\alpha(\phi, \theta))^{2^{s-1}} = \left(\delta_0{}^x \prod \gamma_i^{t_i}\right)^{2^{s-1}} = \delta_0^{x2^{s-1}} = 1.$$

Finally, suppose that G/C is not cyclic, x is even, and $\xi\sigma^{2^t}$ is a square in C. Thus a_i is odd if and only if t_i is odd for $i = 1, 2,\ldots, k$. Now by condition (N8)(b), there must be an even number of j's such that $\gamma_j^{a_j}$ has order 2^s. But $\gamma_j^{a_j}$ has order 2^s if and only if $\gamma_j^{t_j}$ has order 2^s so there are an even number of j's such that $\gamma_j^{t_j}$ has order 2^s. Hence $\prod_{i=1}^{k} \gamma_i^{t_i}$ has order dividing 2^{s-1}. So $(\alpha(\theta, \phi)/\alpha(\phi, \theta))^{2^{s-1}} = 1$.

This completes the proof of the theorem.

COROLLARY 3.3. *Let $\xi = \xi'\tau^{a(q)}$, where ξ' is in* $\mathrm{Gal}(L/K(\zeta))$ *and τ generates* $\mathrm{Gal}(L(\epsilon_q)/L)$. *Then the maximum 2-local index of an element in $S(K)_2$ is* 1 *or* 2, *and is* 2 *only when $\theta(\zeta) = \zeta^{-1}$ and one of the following occurs.*

(1) *$2^{s-2}f(2)$ is odd;*

(2) *G/C is cyclic and ϕ is not a square in C;*

(3) *G/C is not cyclic and x is odd;*

(4) *G/C is not cyclic and ξ', σ^{2^t}, and $\sigma^{2^t}\xi'$ are not squares in* $\mathrm{Gal}(L/K(\zeta))$;

(5) *G/C is not cyclic, $f(2)$ is odd, and σ^{2^t} is not a square in C.*

Proof. The corollary follows directly from Theorem 3.2 in cases (1)–(4). For (5), observe that by picking q so that 2 is not a square modulo q we get that the power of 2 dividing $a(q)$ is the same as that dividing $f(2)$. So for this choice of q, ξ and $\sigma^{2^t}\xi$ are not squares in C.

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